# Algebra Comprehensive Examination

2014

June 27, 2014

You can get maximum 100 marks and 180 minutes!

1. Let $p$ be a prime number, let $\mathbb{F}_p$ be the finite field with $p$ elements and let $G = \mathrm{GL}_2(\mathbb{F}_p)$. Prove that $G$ has only one conjugacy class of order $p$. Also give one representative of this class. Suggestion: Use Jordan forms.

[15 marks]

2. Find all finite groups $G$ such that $\mid \mathrm{Aut}(G) \mid= 1$. Determine the automorphism group $\mathrm{Aut}(S_3)$.

[15 + 10 marks]

3. Let $\mathbb{F}_q$ be a finite field with $q$ elements. A complete flag in the vector space $\mathbb{F}_q^n$ is a nested sequence of linear subspaces $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{F}_q^n$ of dimensions $0, 1, ..., n$ respectively. Let $f_n(q)$ be the number of complete flags in $\mathbb{F}_q^n$ as a rational function of $q$. Find the limit of $f_n(q)$ as $q$ tends to 1.

[10 marks]

4. Let $R$ be a commutative ring with unity.

(i) Let $I$ and $J$ be prime ideals of $R = \mathbb{Z}[\sqrt{2}]$ such that $I \cap \mathbb{Z} = 3\mathbb{Z}$ and $J \cap \mathbb{Z} = 7\mathbb{Z}$. Prove that $R/I \otimes_R R/J = 0$.

(ii) Prove that a $R$-module $M$ is flat if and only if $\mathrm{Tor}_n^R(M, N) = 0$ for all $R$-modules $N$ and $n > 0$.

[10 + 10 marks]

5. Let $f(X)$ be an irreducible polynomial of degree $n$ over a field $F$ and let $g(X) \in F[x]$. Prove that the degree of every irreducible factor of $f(g(X)) \in F[x]$ is divisible by $n$. [15 marks]

6. Let $f$ be an irreducible polynomial of degree 5 in $\mathbb{Q}[X]$. Suppose that in $\mathbb{C}$, $f$ has exactly two nonreal roots. Prove that the Galois group of the splitting field of $f$ is isomorphic to $S_5$. [15 marks]

[Suggestion: Show that $S_5$ is generated by a transposition and a 5-cycle.]

**Comprehensive Examination July, 2014** IISER PUNE

# Differential Equations

Date: 16 July, 2014

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Instructions:

- Total Marks for this question paper is 100.
- Attempt any 6 questions.
- All questions carry equal marks.
- Please state clearly the Theorems or Results you use.

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1. Give two different proofs of the following: Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$. Then there exists at most one solution $u \in C^2(\overline{\Omega})$ of the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega.$$

Ans. Can be proved either as an Application of maximum principle for harmonic functions or using energy methods.

2. Let $f \in C_0^\infty(\mathbb{R}^n \times [0, t_0])$. Let $v \in C_0^\infty(\mathbb{R}^n \times [0, t_0])$ satisfy

$$v_t - \Delta v = f \text{ in } \mathbb{R}^n \times (0, t_0), \qquad v = 0 \text{ on } \mathbb{R}^n \times \{t = 0\}.$$

Let $\tilde{v} := \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)\, f(y, s)\, dy\, ds$. Find $v$.

Ans. Notice that both $v$ and $\tilde{v}$ are bounded functions and hence we apply the Uniqueness for Cauchy problem and get $v = \tilde{v}$.

3. Explain the calculus of variations behind the Euler-Lagrange equations for a smooth Lagrangian. Also, find the Euler-Lagrange equation for the Lagrangian $L$ given by $L(q, x) = \frac{1}{2}m|q|^2 - \phi(x)$, where $m > 0$.

Ans. Take all $\mathcal{C}^2$ curves starting at $y$ at time 0 and reaching the point $x$ at time $t$. A basic question in the calculus of variations is then to find such a curve minimizing an action functional associated to the Lagrangian. The Euler Langrange equation for the given Lagrangian is obtained simply by substituting this $L$ in the the standard expression of the Euler-Lagrange Equation.

4. Consider the following non-linear system of differential equations:

$$\dot{x} = \begin{pmatrix} -x_2 - x_1 x_2^2 + x_3^2 - x_1^3 \\ x_1 + x_3^3 - x_2^3 \\ -x_1 x_3 - x_3 x_1^2 - x_2 x_3^2 - x_3^5 \end{pmatrix}.$$

Use an appropriate Liapunov function to see whether the origin is a stable equilibrium point or asymptotically stable equilibrium point or is unstable. Study the trajectories of the linearized system $\dot{x} = Df(0)x$ for this problem to justify that the origin is stable but not asymptotically stable for the linearized system.

Ans. Use the Liapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$ and show that $\dot{V}(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. This gives that the origin is an asymptotically stable equilibrium point. The matrix for the Linearized system is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The trajectory solves $\dot{x}_3 = 0$ and hence $x_3 =$ constant. Also the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $i, -i$. Hence the trajectories lie on circles in planes parallel to the $x_1, x_2$ plane; hence the origin is stable but not asymptotically stable for the linearized system.

5. Solve the following system of non-linear differential equations

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = -x_2 + x_1^2$$

and show that the stable manifold $S$ and the unstable manifold $U$ near the origin are given by

$$S : x_2 = -\frac{x_1^2}{3} \qquad \text{and} \qquad U : x_1 = 0.$$

Sketch $S, U, E^s$ and $E^u$; where $E^s, E^u$ are the stable and unstable subspaces of the linearised system at the origin respectively.

Ans. First we find three successive approximations $u^1(t,a), u^2(t,a)$ and $u^3(t,a)$ for the system and use $u^3(t,a)$ to approximate $S$ near the origin for this system. the observation that $u^3(t,a) = u^2(t,a)$ implies that $u^{j+1}(t,a) = u^j(t,a)$ for $j \geq 2$. Thus $u(t,a) = u^2(t,a)$ which gives the exact function defining $S$.

The matrix $A$ of the linearized system at the origin is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The solution is given by $x_1(t) = c_1 e^{-t}, x_2(t) = c_2 e^{-t} + c_1^2 \left(e^{-t} - e^{-2t}\right)$ where $c = x(0)$. Clearly $\lim_{t\to\infty} \phi_t(c) = 0$ iff $c_2 = -\frac{c_1^2}{3}$. Thus the expression for $S$. Also, $\lim_{t\to-\infty} \phi_t(c) = 0$ iff $c_1 = 0$. Thus the expression for $U$. $E^s$ and $E^u$ are simpler.

6. For $f \in \mathcal{C}^1(\mathbb{R}^n)$ and for each $x_0 \in \mathbb{R}^n$, the initial value problem

$$\dot{x} = \frac{f(x)}{1 + |f(x)|}, \qquad x(0) = x_0$$

has a unique solution defined for all $t \in \mathbb{R}$. That is, this system defines a dynamical system on $\mathbb{R}^n$. Here, please prove that the maximal interval of existence is the whole of $\mathbb{R}$.

Ans. We know that if $f \in \mathcal{C}^1(\mathbb{R}^n)$ then so is $\frac{f(x)}{1 + |f(x)|}$. Let $x(t)$ be the solution of the given IVP on the maximal interval of existence $(\alpha, \beta)$. Then, we know that $x(t)$ satisfies the integral equation

$$x(t) = x_0 + \int_0^t \frac{f(x(s))}{1 + |f(x(s))|} ds$$

for all $t \in (\alpha, \beta)$. Suppose $\beta < \infty$ then since $|x(t)| \le |x_0| + \beta$ for all $t \in [0, \beta)$, the solution through the point $x_0$ at time $t = 0$ is contained in a compact set. But then, $\beta = \infty$, a contradiction. Therefore, $\beta = \infty$. A similar proof shows that $\alpha = -\infty$.

7. The autonomous ode
$$u' = f(u), \quad u(t_0) = u_0$$
is to be solved by the numerical method
$$u^{n+2} = u^{n+1} + \frac{\Delta t}{12}\left(5f(u^{n+2}) + 8f(u^{n+1}) - f(u^n)\right). \tag{1}$$
Using Taylor expansion to find the local order of the method.

Ans. Assume that $f$ is sufficiently smooth so that the taylor Series expansion of $f$ is valid. (i) Expand $u(t + 2\Delta t)$ and $u(t + \Delta t)$.

$$u(t + 2\Delta t) = u(t) + 2\,\Delta t\, u'(t) + 2\,(\Delta t)^2\, u''(t) + \frac{4}{3}\,(\Delta t)^3\, u'''(t) + O((\Delta t)^4).$$

$$u(t + \Delta t) = u(t) + \Delta t\, u'(t) + \frac{1}{2}\,(\Delta t)^2\, u''(t) + \frac{1}{6}\,(\Delta t)^3\, u'''(t) + O((\Delta t)^4).$$

(ii) We need expansions of $f(u(t + 2\,\Delta t))$ and $f(u(t + \Delta t))$.

$$f(u(t{+}2\Delta t)) = f(u(t)){+}(2\,\Delta t\, u'(t){+}2\,(\Delta t)^2\, u''(t))\, f'(u(t)){+}2\,(\Delta t)^2(u')^2\, f''(u(t)){+}O((\Delta t)^3).$$

$$f(u(t{+}\Delta t)) = f(u(t)){+}(\Delta t\, u'(t){+}\frac{1}{2}\,(\Delta t)^2\, u''(t))\, f'(u(t)){+}\frac{1}{2}\,(\Delta t)^2(u')^2\, f''(u(t)){+}O((\Delta t)^3).$$

(iii) Observe $u' = f(u)$ hence $u'' = f'(u)u'$ and $u''' = f''(u)\,(u')^2 + f'(u)u'' = f''(u)\,(u')^2 + (f'(u))^2u'$. (iv) Finally we plug everything up in the scheme

$$u^{n+2} = u^{n+1} + \frac{\Delta t}{12}\left(5f(u^{n+2}) + 8f(u^{n+1}) - f(u^n)\right)$$

to get

$$u(t) + 2\Delta t\, u'(t) + 2(\Delta t)^2u''(t) + \frac{4}{3}(\Delta t)^3\, u'''(t) + O((\Delta t)^4)$$

$$= u(t) + \Delta t\, u'(t) + \frac{1}{2!}(\Delta t)^2u''(t) + frac16(\Delta t)^3\, u'''(t) + O((\Delta t)^4)$$

$$+\frac{\Delta t}{12}\left(5(f(u(t)) + 2\Delta t u' f'(u) + 2(\Delta t)^2u''(t)f'(u) + 2(\Delta t)^2(u')^2f''(u)\right)$$

$$+8(f(u(t)) + \Delta t u' f'(u) + \frac{1}{2!}(\Delta t)^2u''(t)f'(u) + \frac{1}{2!}(\Delta t)^2(u')^2f''(u)) - f(u))$$

$$= O((\Delta t)^4).$$

Hence local order is $O((\Delta t)^4)$.

*Write your answers in the answer sheets provided. Give full explanation with clear statements of any theorem you use. Use no books or notes in this exam. Attempt all problems. You have 3 hours.*

1. (a) (6 points) Prove the identity:

$$\sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x(x-1)\dots(x-k+1) = x^n.$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denotes the Stirling partition number: the number of partitions of $\{1, 2, \dots, n\}$ into exactly $k$ nonempty subsets.

(b) (4 points) Determine the exponential generating function of the sequence

$$\left\{ \begin{array}{c} 0 \\ k \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ k \end{array} \right\}, \dots, \left\{ \begin{array}{c} k \\ k \end{array} \right\}, \left\{ \begin{array}{c} k+1 \\ k \end{array} \right\}, \dots.$$

2. (10 points) Show that the number of nonequivalent colourings of the corners of a regular 5-gon with $p$ colours is $\frac{p(p^2+4)(p^2+1)}{10}$.

3. (10 points) If $G$ is a graph without isolated vertices, then prove that $\alpha'(G) + \beta'(G) = n$, where $\alpha'(G)$ denotes maximum size of matching, $\beta'(G)$ denotes minimum size of edge cover, and $n$ is the number of vertices of $G$.

4. (10 points) Prove or disprove: If $P$ is a $u, v$-path in a 2-connected graph $G$, then there is a $u, v$-path $Q$ that is internally disjoint from $P$.

5. (a) (5 points) Let $M$ and $M'$ be matchings in a bipartite graph $G = (X \cup Y, E)$. Suppose that $M$ saturates $S \subseteq X$ and that $M'$ saturates $T \subseteq Y$. Prove that $G$ has a matching that saturates $S \cup T$.

   (b) (5 points) Let $k(G)$ and $k'(G)$ denote the vertex-connectivity and edge-connectivity respectively of a graph $G$. Prove that if $G$ is a 3-regular graph then $k(G) = k'(G)$.

6. Consider a tree $T$ on $n$ nodes. We wish to assign distinct numbers in the range $\{1, 2, \dots, n\}$ to the nodes. Such an assignment is called compact if for every $i$, the nodes with labels in the range $\{1, 2, \dots, i\}$ induce a connected subtree of $T$.

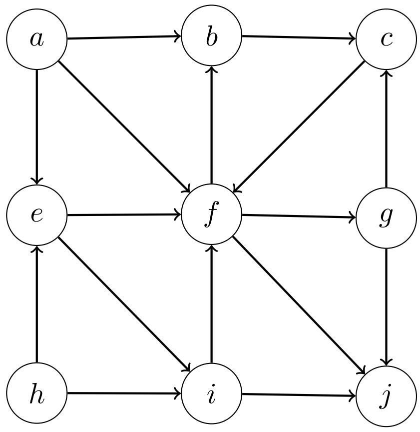
   (a) (10 points) Suppose $T$ is a rooted complete binary tree of depth $d$; in particular, it has $2^d$ leaves. How many compact assignments are there such that the root is assigned the value 1? (Hint: Go recursively down the tree and at each point decide what values will go to the root, what will go to the left subtree and what values will go to the right subtree. To get full credit you should give a compact formula!)

   (b) (9 points) Suppose $T$ is an arbitrary (not necessarily binary) rooted tree and the root has to assigne the value 1. Give a polynomial time algorithm that determines the number of such compact assignments for $T$.

   (c) (6 points) Suppose $T$ is an arbitrary tree and there is no restriction on where the element 1 might appear. Assume that the problem in the previous part (that is, for rooted trees) can be solved in polynomial time. Use this to give a polynomial time algorithm to determine the number of compact assignments for the tree $T$.

7. We are given a directed graph $G = (V, E)$. In addition we are given non-negative integers $a_v$ and $b_v$ for each vertex $v \in V$. We would like to determine a subgraph $G' = (V, E')$ of $G$ with the maximum possible number of edges such that the following in-degree and out-degree constraints are obeyed in $G'$: $\mathsf{in\text{-}degree}(v) \leq a_v$ and $\mathsf{out\text{-}degree}(v) \leq b_v$, for $v \in V$. You have to solve this problem using network flows. First, we construct a bipartite graph $H = (V \cup V', E)$, where $V = V(G)$ and $V' = \{v' : v \in V(G)\}$. That is, we duplicate the vertices of $G$. The edges go from $V$ to $V'$: $(u, v') \in E(H)$ iff $(u, v) \in E(G)$.

(a) (2 points) Consider the example $G$ shown. Draw the bipartite graph $H$ corresponding to this graph.




(b) (3 points) The subgraph $G'$ of $G$ corresponds to a subgraph $H'$ of $H$. Show what constraints $H'$ must satisfy if $G'$ satisfies the degree constraints above.

(c) (7 points) Formulate the problem as a network flows problem. State clearly why maximum flow in this network corresponds to the optimal subgraph $G'$. [Hint: Add a source vertex $s$ to $H$ and edges connecting it to all vertices in $V$, and place a sink $t$ with edges going into it from all vertices is $V'$.] What must be the capacities on the edges of this graph?

(d) (8 points) Now assume that all $a_v$ and $b_v$ are 2 in the above example. Use a network flows algorithm on your network to determine the best subgraph $G'$. Give a succinct reason why no better subgraph than the one you found can exist?

(e) (5 points) State how one should proceed in general when the degree constraints are not uniform. Is the maximum flow then always integral? How would you certify that the solution found is optimal?

**TOPOLOGY COMPREHENSIVE EXAM 2014**

There are SIX questions in this exam. All questions are compulsory. Time: 3 hours. All the best!

(1) (a) Define and give non-trivial examples of each of the following.
  (i) Vector bundle.
  (ii) Connection on a vector bundle.
  (iii) Geodesic.
(b) State the following theorems.
  (i) Seifert Van Kampen theorem.
  (ii) Sard's theorem.
  (iii) Borsuk-Ulam theorem.

(2) Let $f : X \to Y$ be a covering space, where $X$ and $Y$ are path connected, locally path connected topological spaces. Let $x \in X$ be a point and $y = f(x)$. Show that for any $n \geq 2$, $f_* : \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism. Is this true for $n = 1$ (explain using an example) ?

(3) Consider a diagram of maps between topological spaces

$$\begin{tikzcd} E_Z \arrow[r, "g'"] \arrow[d, "f'"'] & E \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

where $E_Z = \{(e, z) \in E \times Z \mid f(e) = g(z)\}$ with topology induced from $E \times Z$ and $f'(e, z) = z$. Show that if $f$ is a covering space then so is $f'$.

(4) (a) Define the chain complex for cellular homology. Clearly give the chain group and boundary map.
(b) Let $A = S^1$, $X_1 = D^2$ and $X_2 = D^2$. Let $\phi : \partial X_1 \to A$ be the map $\phi(z) = z^2$. Let $\psi : \partial X_2 \to A$ be the map $\psi(z) = z^3$. Define $X = X_1 \cup X_2 \cup A/\sim$, where $\sim$ is the equivalence relation given by $z \sim \phi(z)$ for all $z \in \partial X_1$ and $z \sim \psi(z)$ for all $z \in \partial X_2$. Compute all the homology groups of $X$.

(5) Let $X = \mathbb{R}P^2 \times S^1$.
(a) Calculate all the deRham cohomology groups of $X$.
(b) Use the Lefschetz fixed point theorem to show that the Euler characteristic of $X$ is zero.

(6) (a) Prove that $S^2$ is not a Lie Group with respect to any group operation. (Hint: Use the Tangent bundle $TS^2$.)

(b) Let $T^2 = [0, 2\pi] \times [0, 2\pi]/\sim$ where $(0, y) \sim (2\pi, y), (x, 0) \sim (x, 2\pi)$. Define the 1-form $\omega = \sin(x)\cos(y)dx + \sin(y)dy$ on $T^2$. Calculate $d\omega_p(v, w)$ i.e., $d\omega$ acting on the vectors $v$, $w$ at the point $p$; where $v = (1, 2)$, $w = (0, 1)$ with respect to the frame $\{\partial/\partial x,\ \partial/\partial y\}$ and $p = (\frac{\pi}{2}, \frac{\pi}{2})$.